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Remarks on the Mean-Field Theory Based on the SO(2N+1) Lie Algebra of the Fermion Operators

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Dedicated to the Memory of Hideo Fukutome

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- **Introduction; Motivation**

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- **Generalized Hartree-Bogoliubov mean-field (MF) Hamiltonian and its diagonalization**

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- **References**

- **Mean-field theory based on the $\mathfrak{Jacobi}\ \mathfrak{hsp}$:= semi-direct sum $\mathfrak{h}_N \rtimes \mathfrak{sp}(2N,\mathbb{R})_{\mathbb{C}}$ algebra of boson operators,**
  S. N. & J. da P., J. Math. Phys. 60 (2019) 081706,

- **Remarks on the MFT based on the SO(2N+1) Lie algebra of the fermion operators,**
  S. N. & J. da P., Int. J. Geom. Methods Mod. Phys. 16 (2019) 1950184,

- **Time dependent $SO(2N+1)$ theory for unified description of bose and fermi type collective excitations,**
  H. F. & S. N., Progr. Theor. Phys. 72 (1984) 239.

§1. Introduction, Motivation:

A theory for self-consistent field (SCF) description of Fermi collective excitations has been proposed, based on the $SO(2N+1)$ Lie algebra of fermion operators;

Fermion mean-field theory(MFT) on Kähler coset space $\frac{G}{H} = \frac{SO(2N+2)}{U(N+1)}$ is given, basing on $SO(2N+1)$ Lie algebra of fermion operators. Embedding $SO(2N+1)$ group into $SO(2N+2)$ group and using $\frac{SO(2N+2)}{U(N+1)}$ coset variables, we give the new MFT on the symmetric space $\frac{SO(2N+2)}{U(N+1)}$;

1) We take an Hamiltonian consisting of the generalized HB (GHB) MF Hamiltonian (MFH) and also assume a linear MFH expressed in terms of the generators of the $SO(2N+1)$ Lie algebra.

2) Diagonalizing MFH, a new aspect of eigenvalues of MFH is shown. Excitation energy arisen from additional SCF parameter, never been seen in the traditional fermion MFT, is derived.

- **S. Berceanu, L. Boutet de Monvel and A. Gheorghe**

  Linear dynamical systems, coherent state manifolds, flows, and matrix Riccati equation, *J. Math. Phys.* **34** (1992) 2353-2371;

  On equations of motion on compact Hermitian symmetric spaces, *J. Math. Phys.* **33** (1992) 998-1007;

§2. SO(2N+1) Bogoliubov transformation and GDM

We consider a fermion system with N single-particle states. Let c_α and $c_\alpha^\dagger$ $(\alpha=1,\cdots,N)$ be the annihilation-creation operatorss satisfying the canonical commutation relation for the fermion.

$$\boxed{\{c_\alpha, c_\beta^\dagger\}=\delta_{\alpha\beta}, \{c_\alpha, c_\beta\}=\{c_\alpha^\dagger, c_\beta^\dagger\}=0.}$$

$$\boxed{c_\alpha,\ c_\alpha^\dagger,\quad E^\alpha{}_\beta=c_\alpha^\dagger c_\beta-\frac{1}{2}\delta_{\alpha\beta}=E^{\beta\dagger}_{\ \alpha}}$$

,

$$\boxed{E^{\alpha\beta}=c_\alpha^\dagger c_\beta^\dagger=E_{\beta\alpha}^\dagger,\quad E_{\alpha\beta}=c_\alpha c_\beta=-E_{\beta\alpha}, I,}$$

which are identified with generators of the Lie algebra $SO(2N+1)$.

The $SO(2N+1)$ Lie algebra of the fermion operators contains $U(N)(=\{E^\alpha{}_\beta\})$ as sub-algebra.

The operator $(-1)^n : n=c_\alpha^\dagger c_\alpha$ anti-commutes with c_α and $c_\alpha^\dagger$,

$$\boxed{\{c_\alpha,\ (-1)^n\}=\{c_\alpha^\dagger,\ (-1)^n\}=0.}$$

Introduce operator $\Theta \equiv \theta_\alpha c_\alpha^\dagger - \overline{\theta}_\alpha c_\alpha, \ \Theta^2 = -\overline{\theta}_\alpha \theta_\alpha \equiv -\theta^2,$
then we have

$$e^\Theta = Z + X_\alpha c_\alpha^\dagger - \overline{X}_\alpha c_\alpha, \overline{X}_\alpha X_\alpha + Z^2 = 1, Z = \cos\theta, X_\alpha = \frac{\theta_\alpha}{\theta}\sin\theta. \tag{1}$$

We obtain,

$$\left.\begin{aligned} & e^\Theta (c, c^\dagger, \frac{1}{\sqrt{2}})(-1)^n e^{-\Theta} = (c, c^\dagger, \frac{1}{\sqrt{2}})(-1)^n G_X, \\ & G_X \stackrel{\mathrm{def}}{=} \begin{bmatrix} I_N - \overline{X}X^{\mathrm{T}} & \overline{X}X^\dagger & -\sqrt{2}Z\overline{X} \\ XX^{\mathrm{T}} & I_N - XX^\dagger & \sqrt{2}ZX \\ \sqrt{2}ZX^{\mathrm{T}} & -\sqrt{2}ZX^\dagger & 2Z^2 - 1 \end{bmatrix}. \end{aligned}\right\} \tag{2}$$

Let G be the $(2N+1)\times(2N+1)$ matrix defined by

$$G \equiv G_X \begin{bmatrix} a & \bar{b} & 0 \\ b & \bar{a} & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} a - \overline{X}Y & \bar{b} + \overline{XY} & -\sqrt{2}Z\overline{X} \\ b + XY & \bar{a} - X\overline{Y} & \sqrt{2}ZX \\ \sqrt{2}ZY & -\sqrt{2}Z\overline{Y} & 2Z^2 - 1 \end{bmatrix}, \tag{3}$$

$$X = \bar{a}Y^{\mathrm{T}} - bY^\dagger,$$

$$Y = X^{\mathrm{T}}a - X^\dagger b,$$

$$YY^\dagger + Z^2 = 1.$$

X: column, Y: row, vectors. The $SO(2N+1)$ canonical transformation (TR) $U(G)$ is generated by the fermion $SO(2N+1)$ Lie operators. The $U(G)$ is extension of the generalized Bogoliubov TR $U(g)$ to a **nonlinear Bogoliubov TR**.

By the TR $U(G)$ for fermion $[c, c^\dagger, \frac{1}{\sqrt{2}}]$, we obtain

$$U(G)\left[c,\ c^\dagger, \frac{1}{\sqrt{2}}\right](-1)^n U^{-1}(G) = \left[c,\ c^\dagger, \frac{1}{\sqrt{2}}\right](-1)^n G,$$

$$G \stackrel{\mathrm{def}}{=} \begin{bmatrix} A & \overline{B} & -\frac{\overline{x}}{\sqrt{2}} \\ B & \overline{A} & \frac{x}{\sqrt{2}} \\ \frac{y}{\sqrt{2}} & -\frac{\overline{y}}{\sqrt{2}} & z \end{bmatrix}, |G>=U(G)|0>, g \stackrel{\mathrm{def}}{=} \begin{bmatrix} a & \overline{b} \\ b & \overline{a} \end{bmatrix}, |g>=U(g)|0>. \quad (4)$$

$|G>/|g>$ are the $SO(2N+1)/SO(2N)$ coherent states, respectively.

$N \times N$ matrices $A = (A^\alpha{}_\beta)$ and $B = (B_{\alpha\beta})$ and N-dimensional column and row vectors $x = (x_\alpha)$ and $y = (y_i)$ and z are defined as

$$A \equiv a - \overline{X}Y = a - \frac{\overline{x}y}{2(1+z)},\ B \equiv b + XY = b + \frac{xy}{2(1+z)},$$
$$x \equiv 2ZX,\ y \equiv 2ZY,\ z \equiv 2Z^2 - 1. \quad (5)$$

Using $U(G)(c,c^\dagger,\frac{1}{\sqrt{2}})U^\dagger(G) = U(G)(c,c^\dagger,\frac{1}{\sqrt{2}})U^\dagger(G)(z+\rho)(-1)^n$, Eq. (5) can be written with a q-number gauge factor $(z-\rho)$ as

$$U(G)(c,c^\dagger,\frac{1}{\sqrt{2}})U^\dagger(G) = (c,c^\dagger,\frac{1}{\sqrt{2}})(z-\rho)G, \tag{6}$$

$$G^\dagger G = GG^\dagger = 1_{2N+1}, \det G = 1,$$

$$U(G)U(G') = U(GG'), U(G^{-1}) = U^{-1}(G) = U^\dagger(G), U(1_{2N+1}) = \mathbb{I}_G, \tag{7}$$

The $U(G)$ is the **nonlinear TR** with a q-number gauge factor $(z-\rho)$ where ρ is given as $\rho = x_\alpha c^\dagger_\alpha - \overline{x}_\alpha c_\alpha$ and $\rho^2 = -\overline{x}_\alpha x_\alpha = z^2 - 1$.

$SO(2N+1)$ ***GDM***

We consider the following $SO(2N+1)$ GDM:

$$\mathcal{W} \stackrel{\mathrm{def}}{=} G \begin{bmatrix} -1_N & 0 & 0 \\ 0 & 1_N & 0 \\ 0 & 0 & 1 \end{bmatrix} G^\dagger, \quad \mathcal{W}^\dagger = \mathcal{W}, \quad \mathcal{W}^2 = 1_{2N+1}. \tag{8}$$

Using $\mathcal{W}$, we will attempt a different approach to the derivation of the unified $SO(2N+1)$ HB eigenvalue equation (EE) from the fermion MF Hamiltonian.

§3. GHB mean-field Hamiltonian and its diagonalization

GHB MFH for which we assume a linear MFH expressed in terms of the generators of the $SO(2N+1)$ algebra:

$$\boxed{\begin{aligned}H_{SO(2N+1)} = F_{\alpha\beta}\left(c_{\alpha}^{\dagger}c_{\beta}-\frac{1}{2}\delta_{\alpha\beta}\right)+\frac{1}{2}D_{\alpha\beta}c_{\alpha}c_{\beta} - \frac{1}{2}\overline{D}_{\alpha\beta}c_{\alpha}^{\dagger}c_{\beta}^{\dagger}\\ + M_{\alpha}c_{\alpha}^{\dagger} + \overline{M}_{\alpha}c_{\alpha},\end{aligned}} \quad (9)$$

$$H_{SO(2N+1)}=\frac{1}{2}\left[c,\ c^{\dagger}, \frac{1}{\sqrt{2}}\right]\mathring{\mathcal{F}}_{0}\begin{bmatrix} c^{\dagger}, \\ c, \\ \frac{1}{\sqrt{2}} \end{bmatrix},\ \mathcal{F}_{g}\equiv\begin{bmatrix} F_{g} & D_{g} \\ -\overline{D}_{g} & -\overline{F}_{g} \end{bmatrix}. \quad (10)$$

The $\mathring{\mathcal{F}}_{0}$ is given by

$$\mathring{\mathcal{F}}\equiv\begin{bmatrix} F_{g} & D_{g} & \sqrt{2}M \\ -\overline{D}_{g} & -\overline{F}_{g} & \sqrt{2}\,\overline{M} \\ \sqrt{2}M^{\dagger} & \sqrt{2}M^{\mathrm{T}} & 0 \end{bmatrix}, \mathring{\mathcal{F}}\equiv\begin{bmatrix} 0 & 1_{N} & 0 \\ 1_{N} & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\mathring{\mathcal{F}}_{0}\begin{bmatrix} 0 & 1_{N} & 0 \\ 1_{N} & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \quad (11)$$

We diagonalize the MFH $H_{SO(2N+1)}$ as follows:

$$H_{SO(2N+1)}=\frac{1}{2}\left[d, d^{\dagger}, \frac{1}{\sqrt{2}}\right]G^{\dagger}\mathring{\mathcal{F}}_{0}G\begin{bmatrix} d^{\dagger}, \\ d, \\ \frac{1}{\sqrt{2}} \end{bmatrix}, G^{\dagger}\mathring{\mathcal{F}}_{0}G=\begin{bmatrix} E_{2N}\cdot 1_{2N} & 0 \\ 0 & 0 \end{bmatrix}, \quad (12)$$

where $E_{2N}=\left[E_{\mathrm{diag.}}, E_{\mathrm{diag.}}\right]$, $E_{\mathrm{diag.}}\equiv[E_{1},\cdots,E_{N}]$. E_{i} is a quasi-particle energy.

SCF condition

If the conditions $\underline{FD - D\overline{F} = 0}$ and $\underline{bee y^\dagger - \overline{a} ee y^{\mathrm{T}} = 0}$ are satisfied, then we have the expressions for x and $x^\dagger$ as

$$\boxed{\begin{aligned} \frac{x}{\sqrt{2}} &= -\left(FF^\dagger + DD^\dagger\right)^{-1}\left(F\sqrt{2}zM + D\sqrt{2}z\overline{M}\right), \\ \frac{x^\dagger}{\sqrt{2}} &= -\left(\sqrt{2}zM^\dagger F^\dagger + \sqrt{2}zM^{\mathrm{T}}D^\dagger\right)\left(FF^\dagger + DD^\dagger\right)^{-1}. \end{aligned}} \tag{13}$$

Then, at last we could reach the expressions for $\frac{x}{\sqrt{2}}$ and $\frac{x^\dagger}{\sqrt{2}}$:

$$\left.\boxed{\begin{aligned} \frac{x}{\sqrt{2}} &= \left(FF^\dagger + DD^\dagger\right)^{-1}\frac{2z^2}{1-z^2}\sqrt{2}M\frac{y}{\sqrt{2}}e\frac{y^\dagger}{\sqrt{2}} \\ &\approx \frac{2z^2}{1-z^2} <e> \left(FF^\dagger + DD^\dagger\right)^{-1}\sqrt{2}M, \\ \frac{x^\dagger}{\sqrt{2}} &= \frac{2z^2}{1-z^2}\frac{y}{\sqrt{2}}e\frac{y^\dagger}{\sqrt{2}}\sqrt{2}M^\dagger\left(FF^\dagger + DD^\dagger\right)^{-1} \\ &\approx \frac{2z^2}{1-z^2} <e> \sqrt{2}M^\dagger\left(FF^\dagger + DD^\dagger\right)^{-1}. \end{aligned}}\right\} \tag{14}$$

$<e> \equiv \frac{y}{\sqrt{2}}e\frac{y^\dagger}{\sqrt{2}}$ means the **averaged eigenvalue distribution**. This is the first time that the final solutions for $\frac{x}{\sqrt{2}}$ and $\frac{x^\dagger}{\sqrt{2}}$ could be derived within the present framework of the $SO(2N+1)$ MFT. It takes place also for the Jacobi-algebra MFT for a boson system.

The **inner product** of the vectors leads to the relation:

$$\boxed{\frac{x^\dagger}{\sqrt{2}}\frac{x}{\sqrt{2}} = \frac{1-z^2}{2} = \frac{4z^4}{(1-z^2)^2} 2 < e >^2 M^\dagger \left(FF^\dagger + DD^\dagger\right)^{-2} M,}$$

which is an appreciably interesting result in the $SO(2N+1)$ MFT and simply rewritten as

$$\boxed{\frac{16z^4}{(1-z^2)^3} < e >^2 M^\dagger \left(FF^\dagger + DD^\dagger\right)^{-2} M = 1.} \tag{15}$$

The relation (15) designates that the additional SCF parameters Ms are inevitably restricted by the behavior of SCF parameters F, D governed by the condition $\underline{FD - D\overline{F} = 0}$.

Remember that <u>this condition is one of the crucial condition</u> to derive the equations for vectors $\frac{x}{\sqrt{2}}$ and $\frac{x^\dagger}{\sqrt{2}}$ which reflect the special aspect of the $SO(2N+1)$ MFT. **Such a result should not be a surprised consequence** that the relation (15) is very similar to the relation obtained in the boson GHB-MFT. This is because we have adopted the same manner of mathematical computation as the manner that is done for the boson system.

§4. MF approach by another form of GDM

Introducing a matrix g_x represented by

$$
g_x=\begin{bmatrix}
1_N-\frac{1}{\sqrt{1+z}}\frac{\overline{x}}{\sqrt{2}}\frac{1}{\sqrt{1+z}}\frac{x^{\mathrm{T}}}{\sqrt{2}} & \frac{1}{\sqrt{1+z}}\frac{\overline{x}}{\sqrt{2}}\frac{1}{\sqrt{1+z}}\frac{x^{\dagger}}{\sqrt{2}} \\
\frac{1}{\sqrt{1+z}}\frac{x}{\sqrt{2}}\frac{1}{\sqrt{1+z}}\frac{x^{\mathrm{T}}}{\sqrt{2}} & 1_N-\frac{1}{\sqrt{1+z}}\frac{x}{\sqrt{2}}\frac{1}{\sqrt{1+z}}\frac{x^{\dagger}}{\sqrt{2}}
\end{bmatrix},\ g_x^{\dagger}=g_x,
$$

the explicit expression for $\not{\mathcal{W}}$ is given as

$$
\begin{aligned}
\not{\mathcal{W}}&=\begin{bmatrix}
 & & -\frac{\overline{x}}{\sqrt{2}} \\
 & g_x & \\
 & & \frac{x}{\sqrt{2}} \\
\frac{x^{\mathrm{T}}}{\sqrt{2}} & -\frac{x^{\dagger}}{\sqrt{2}} & z
\end{bmatrix}
\begin{bmatrix}
 & & 0 \\
 & \not{W} & \\
 & & 0 \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
 & & \frac{\overline{x}}{\sqrt{2}} \\
 & g_x^{\dagger} & \\
 & & -\frac{x}{\sqrt{2}} \\
-\frac{x^{\mathrm{T}}}{\sqrt{2}} & \frac{x^{\dagger}}{\sqrt{2}} & z
\end{bmatrix},\ \not{W}\equiv g\begin{bmatrix}-1_N & 0 \\ 0 & 1_N\end{bmatrix}g^{\dagger}, \qquad (16)\\
&=\begin{bmatrix}
g_x\not{W}g_x^{\dagger} & g_x\not{W}\begin{bmatrix}\frac{\overline{x}}{\sqrt{2}} \\ -\frac{x}{\sqrt{2}}\end{bmatrix} \\
\begin{bmatrix}\frac{x^{\mathrm{T}}}{\sqrt{2}} & -\frac{x^{\dagger}}{\sqrt{2}}\end{bmatrix}\not{W}g_x^{\dagger} & z^2
\end{bmatrix}
+\begin{bmatrix}
\frac{\overline{x}}{\sqrt{2}}\frac{x^{\mathrm{T}}}{\sqrt{2}} & 0 & -z\frac{\overline{x}}{\sqrt{2}} \\
0 & \frac{x}{\sqrt{2}}\frac{x^{\dagger}}{\sqrt{2}} & z\frac{x}{\sqrt{2}} \\
-z\frac{x^{\mathrm{T}}}{\sqrt{2}} & z\frac{x^{\dagger}}{\sqrt{2}} & 0
\end{bmatrix},
\end{aligned}
$$

where $g_x \not{\mathcal{W}} g_x^\dagger$ is given by

$$
g_x \not{\mathcal{W}} g_x^\dagger = \begin{bmatrix} 2\rho_G - 1_N & -2\overline{\kappa}_G \\ 2\kappa_G & -2\overline{\rho}_G + 1_N \end{bmatrix}, \quad
\begin{aligned}
\rho_G &= R_g - \overline{L} \frac{1}{\sqrt{1+z}} \frac{x^{\mathrm{T}}}{\sqrt{2}} - \frac{1}{\sqrt{1+z}} \frac{\overline{x}}{\sqrt{2}} L^{\mathrm{T}}, \\
\kappa_G &= K_g - L \frac{1}{\sqrt{1+z}} \frac{x^{\mathrm{T}}}{\sqrt{2}} + \frac{1}{\sqrt{1+z}} \frac{x}{\sqrt{2}} L^{\mathrm{T}}.
\end{aligned} \tag{17}
$$

Thus we reach our desired goal: $SO(2N+1)$ GDM $\not{\mathcal{W}}$

$$
\not{\mathcal{W}} = \begin{bmatrix} 2\rho_G - 1_N & -2\overline{\kappa}_G & 2\sqrt{1+z}\,\overline{L} \\ 2\kappa_G & -2\overline{\rho}_G + 1_N & 2\sqrt{1+z}L \\ 2\sqrt{1+z}L^{\mathrm{T}} & 2\sqrt{1+z}L^\dagger & z^2 \end{bmatrix}
+ \begin{bmatrix} \frac{\overline{x}}{\sqrt{2}}\frac{x^{\mathrm{T}}}{\sqrt{2}} & 0 & -z\frac{\overline{x}}{\sqrt{2}} \\ 0 & \frac{x}{\sqrt{2}}\frac{x^\dagger}{\sqrt{2}} & z\frac{x}{\sqrt{2}} \\ -z\frac{x^{\mathrm{T}}}{\sqrt{2}} & z\frac{x^\dagger}{\sqrt{2}} & 0 \end{bmatrix}. \tag{18}
$$

$$
L \equiv \frac{1}{\sqrt{2}} \frac{1}{\sqrt{1+z}} \left(\overline{R}_g x + K_g \overline{x} - \frac{1}{2} x \right). \tag{19}
$$

The GHB MF operator $\mathcal{F}_g$ is transformed to $\not{\mathcal{F}}_G$ as

$$
\not{\mathcal{F}}_G = \left[\begin{array}{cc|c} \multicolumn{2}{c|}{\multirow{2}{*}{g_x}} & -\frac{\overline{x}}{\sqrt{2}} \\ & & \frac{x}{\sqrt{2}} \\ \hline \frac{x^{\mathrm{T}}}{\sqrt{2}} & -\frac{x^\dagger}{\sqrt{2}} & z \end{array} \right]
\left[\begin{array}{cc|c} \multicolumn{2}{c|}{\multirow{2}{*}{$\mathcal{F}_G$}} & \sqrt{2}M \\ & & \sqrt{2}\,\overline{M} \\ \hline \sqrt{2}M^\dagger & \sqrt{2}M^{\mathrm{T}} & 0 \end{array} \right]
\left[\begin{array}{cc|c} \multicolumn{2}{c|}{\multirow{2}{*}{$g_x^\dagger$}} & \frac{\overline{x}}{\sqrt{2}} \\ & & -\frac{x}{\sqrt{2}} \\ \hline -\frac{x^{\mathrm{T}}}{\sqrt{2}} & \frac{x^\dagger}{\sqrt{2}} & z \end{array} \right], \tag{20}
$$

in which, here we use a matrix $\mathcal{F}_G$ which modifies $\mathcal{F}_g$ as

$$
\mathcal{F}_G = \begin{bmatrix} F_G & D_G \\ -\overline{D_G} & -\overline{F_G} \end{bmatrix}, \quad
\begin{aligned}
F_{G\alpha\beta} &= h_{\alpha\beta} + [\alpha\beta|\gamma\delta]\rho_{G\gamma\delta}, \\
D_{G\alpha\beta} &= \frac{1}{2}[\alpha\gamma|\beta\delta]\left(-\kappa_{G\delta\gamma}\right).
\end{aligned} \tag{21}
$$

The transformed MF operator $\not{\mathcal{F}}_G$ is rewritten as $\not{\mathcal{F}}_G =$

$$
\begin{bmatrix}
g_x\mathcal{F}_G g_x^\dagger + g_x \begin{bmatrix} -Mx^{\mathrm{T}} & Mx^\dagger \\ -\overline{M}x^{\mathrm{T}} & \overline{M}x^\dagger \end{bmatrix} + \begin{bmatrix} -\overline{x}M^\dagger & -\overline{x}M^{\mathrm{T}} \\ xM^\dagger & xM^{\mathrm{T}} \end{bmatrix} g_x^\dagger & [g_x\mathcal{F}_G + M^{\mathrm{T}}x - M^\dagger\overline{x}] \begin{bmatrix} \frac{\overline{x}}{\sqrt{2}} \\ -\frac{x}{\sqrt{2}} \end{bmatrix} + \sqrt{2}z g_x \begin{bmatrix} M \\ \overline{M} \end{bmatrix} \\
\begin{bmatrix} \frac{x^{\mathrm{T}}}{\sqrt{2}} & -\frac{x^\dagger}{\sqrt{2}} \end{bmatrix} \left[\mathcal{F}_G g_x^\dagger + \begin{bmatrix} -Mx^{\mathrm{T}} & Mx^\dagger \\ -\overline{M}x^{\mathrm{T}} & \overline{M}x^\dagger \end{bmatrix} \right] + \sqrt{2}z[M^\dagger \; M^{\mathrm{T}}] g_x^\dagger & \begin{bmatrix} \frac{x^{\mathrm{T}}}{\sqrt{2}} & -\frac{x^\dagger}{\sqrt{2}} \end{bmatrix} \mathcal{F}_G \begin{bmatrix} \frac{\overline{x}}{\sqrt{2}} \\ -\frac{x}{\sqrt{2}} \end{bmatrix}
\end{bmatrix} \tag{22}
$$

Thus we reach our final goal: Modified $SO(2N+1)$ HB EE with $\mathcal{F}_G$

$$
\left.
\begin{aligned}
&\left\{ \mathcal{F}_G - 2(1-z)\mathcal{F}_g - \begin{bmatrix} \overline{x}M^\dagger + Mx^{\mathrm{T}} & \overline{x}M^{\mathrm{T}} - Mx^\dagger \\ -\overline{(\overline{x}M^{\mathrm{T}} - Mx^\dagger)} & -\overline{(\overline{x}M^\dagger + Mx^{\mathrm{T}})} \end{bmatrix} \right\} \begin{bmatrix} a \\ b \end{bmatrix}_i = \varepsilon_i \begin{bmatrix} a \\ b \end{bmatrix}_i, \\
&\mathcal{F}_G \begin{bmatrix} \frac{\overline{x}}{\sqrt{2}} \\ -\frac{x}{\sqrt{2}} \end{bmatrix} + z \begin{bmatrix} \sqrt{2}M \\ \sqrt{2}\overline{M} \end{bmatrix} = 0, \; M^{\mathrm{T}}x - M^\dagger\overline{x} = 0, \; i : \underline{\text{quasi-particle state.}}
\end{aligned}
\right\} \tag{23}
$$

§5. *Discussions, perspective and summary*

- **The present MFT relates deeply to the algebraic MFT by Rosensteel based on the coadjoint orbit method.**

Mean field theory for U(n) dynamical groups, *J. Phys. A:Math. Theor.* **44 (2011) 165201:**

There is no necessity to consider only the orbit of determinants, i.e., S-det.

- **For this aim, concept of symplectic structure is useful. This is made to construct a non-degenerate symplectic form ω as an antisymmetric form which is defined on the pair of tangent vector at $\mathcal{G}\mathcal{W}\mathcal{G}^{-1}$,**

$$\omega_{\mathcal{G}\mathcal{W}\mathcal{G}^{-1}}(X,\ Y) \equiv -i\mathbf{tr}\left(\mathcal{G}\mathcal{W}\mathcal{G}^{-1}[X,\ Y]\right). \tag{24}$$

The X and $Y \in SO(2N+2)$ are tangent vectors at $\mathcal{G}\mathcal{W}\mathcal{G}^{-1}$. The idempotent GDM $\mathcal{W}$ forms an orbit surface in the space of all the GDMs.

- **It is necessary to introduce even-dimensional GDM on the $SO(2N+2)$ CS rep. We use the $(2N+2)\times(2N+2)$ HB GDM $\mathcal{W}(\mathcal{W}^2 = \mathcal{W})$. This HB GDM on the $SO(2N+2)$ CS rep is an element of the dual space $\mathcal{G}^*$ of the Lie algebra $SO(2N+2)$. We prepare both the HB GDM $\mathcal{W}$ and its coadjoint orbit $O_{\mathcal{W}} = \{\mathcal{G}\mathcal{W}\mathcal{G}^{-1} | \mathcal{G} \in SO(2N+2)\}$.**

- For the geometrical picture of $O_{\mathcal{W}}$, see Figuire 1. The determinantal orbit is regarded as a symplectic manifold. The $SO(2N+2)$ TR for determinantal orbit $O_{\mathcal{W}}$ preserves the symplectic structure

$$\omega_{\mathcal{W}}(X,Y)=\omega_{\mathcal{G}\mathcal{W}\mathcal{G}^{-1}}(\mathbf{ad}_{\mathcal{G}}(X)\equiv\mathcal{G}X\mathcal{G}^{-1},\mathbf{ad}_{\mathcal{G}}(Y)\equiv\mathcal{G}Y\mathcal{G}^{-1}). \quad (25)$$

The $X,Y\in SO(2N+2)$ are tangent vectors at $\mathcal{W}$. We also define the coadjoint action $\mathbf{Ad}^*_{\mathcal{G}}$ on the GDM on the $SO(2N+2)$ CS rep as $\mathbf{Ad}^*_{\mathcal{G}}(\mathcal{W})=\mathcal{G}\mathcal{W}\mathcal{G}^{-1}$.

- The orbit surface $O_{\mathcal{W}}$ has one-to-one correspondence with the coset space of the $SO(2N+2)$ modulo: The isotropy sub-group arises at $\mathcal{W}$, $\mathcal{H}_{\mathcal{W}}=\{h\in SO(2N+2)|\ h\mathcal{W}h^{-1}=\mathcal{W}\}$ and the coset space is identified with $O_{\mathcal{W}}$, i.e., $\frac{SO(2N+2)}{\mathcal{H}_{\mathcal{W}}}\rightarrow O_{\mathcal{W}}, \mathcal{G}\mathcal{H}_{\mathcal{W}}\rightarrow\mathcal{G}\mathcal{W}\mathcal{G}^{-1}$. In the generic orbit, map $U(\mathcal{G})\Phi\rightarrow\mathcal{G}\mathcal{W}\mathcal{G}^{-1}$ is many-to one correspondence. Ambiguity in the correspondence can be expressed best in terms of the differing isotropy sub-groups.

- We adopt a model Hamiltonian $\widehat{H}$ on the $SO(2N+2)$ and energy function (EF) $H_{\mathcal{W}}(\mathcal{G}\mathcal{W}\mathcal{G}^{-1})\equiv<U(\mathcal{G})\Phi|\widehat{H}_{SO(2N+2)}U(\mathcal{G})\Phi>$. To remove the ambiguity, we have a possibility to choice for the EF by averaging the energy as follows: $\mathcal{H}(\mathcal{G}\mathcal{W}\mathcal{G}^{-1})=\mathbf{min}_{h\in H_{\mathcal{W}}}<U(\mathcal{G})U(h)\Phi|H_{SO(2N+2)}U(\mathcal{G})U(h)\Phi>$. Minimization of EF is made on the orbit surface $O_{\mathcal{W}}$.

- **The HF Hamiltonian $\underline{H_{HF}}$ is the <u>projection of the vector field onto the surface</u> relative to the non-degenerate symplectic form. The <u>TDHF solutions</u> are the <u>integral curves</u> of the HF vector field:**

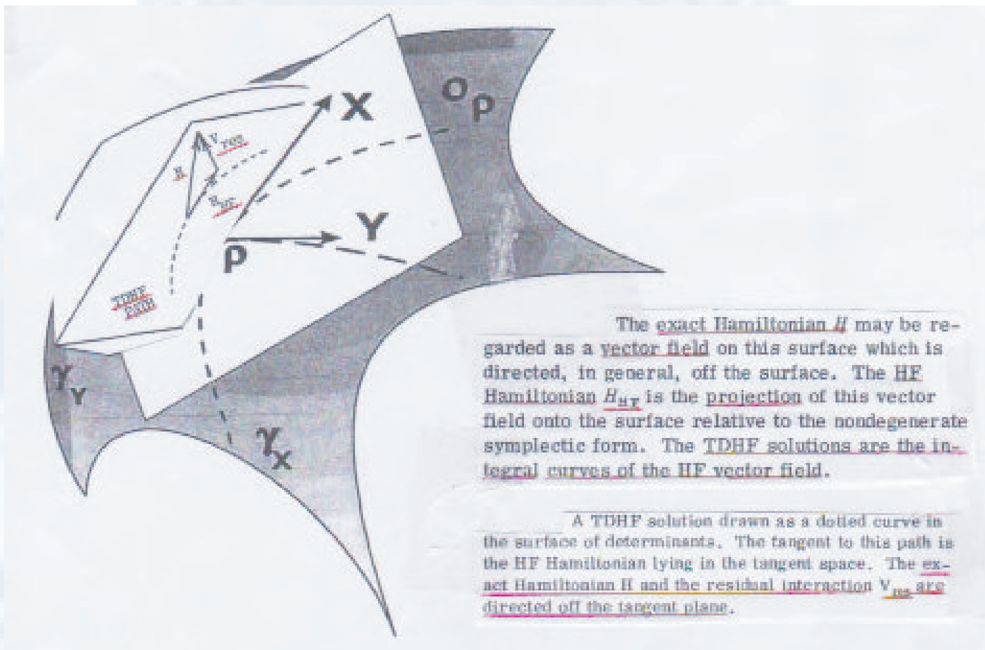

Figure 1. The Lie algebra elements X and Y are geometrically viewed as tangent vectors to the curves γ_X and γ_Y in the coadjoint orbit surface $\mathcal{O}_\rho$.

The Lie algebra elements X and Y are geometrically viewed as tangent vectors to the curves γ_X and γ_Y, respectively, in the coadjoint-orbit surface $\mathcal{O}_\rho$.

- **Finally, we say, we have diagonalized the GHB-MFH and obtained the unpaired-mode amplitudes $|x|^2$ expressed in terms of the SCF and additional SCF parameters and the $SO(2N+1)$ parameter z^2. We have made clear a new aspect of these results which have never been in the traditional works.**